Math 206B Lecture 9 Notes

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1 The RSK Algorithm

1.1 Semi-standard Young tablaeux

We will extend the RS-algorithm, which gave a bijection $\Phi : S_n \to \coprod_{|\lambda|=n} \mathrm{SYT}(\lambda)^2$. RSK will give a bijection $\Phi : M(\overline{a}, \overline{b}) \to \coprod_{|\lambda|=N} \mathrm{SSYT}(\lambda, \overline{a}) \times \mathrm{SSYT}(\lambda, \overline{b})$. Here, $\overline{a} = (a_1, \dots, a_n)$, $\overline{b} = (b_1, \dots, b_n)$, and $N = |\overline{a}| = |\overline{b}| = a_1 + \cdots + a_n = b_1, \dots, b_n$. $M(\overline{a}, \overline{b})$ is the set of $\mathbb{N}^+$ $n \times n$ matrices with row sums $a_1, \dots, a_n$ and column sums $b_1, \dots, b_n$. $\mathrm{SSYT}(\lambda, \overline{a})$ is the set of semi-standard Young tableaux of shape λ and weight $\overline{a}$.

Definition 1.1. A **semi-standard Young tableau** of shape λ is a Young tableau where we are allowed to have numbers reused, and we have numbers are weakly increasing as we go to the right. The **weight** of a semi-standard Young tableau is $(m_1, m_2, \dots)$, where m_i is the number of is in A.

Example 1.1. If $\overline{a} = \overline{e} = (1, \dots, 1)$, then $\mathrm{SSYT}(\lambda, \overline{a}) = \mathrm{SYT}(\lambda)$. Then $M(\overline{a}, \mathbf{a})$ is the number of 0-1 matrices with ros sums equal to 1 and column sums equal to 1. So the number of such matrices is $|S_n| = n!$. This special case is the case of R-S.

Example 1.2. Let $n = 2$, and $\overline{a} = (m, m) = \overline{b}$. Then $\#M(\overline{a}, \overline{b}) = m+1$ because the entire matrix is determined by the upper left entry:

$$\begin{bmatrix} * & \\ & \end{bmatrix}$$

The right hand side is $\coprod_{|\lambda|=2m} \mathrm{SSYT}(\lambda, \overline{a})^2$. Note that $\lambda = (\lambda_1, \lambda_2)$. Otherwise, $\mathrm{SSYT}(\lambda, \overline{a}) = 0$. The number of such λ is $m + 1$:

1	1	$\cdots$	1	1	1
2	$\cdots$	2			

This case is very different from the R-S case.

1.2 Description of the algorithm

Given $M \in M(\bar{a}, \bar{b})$, we first need to turn M into a word.

Example 1.3. Here is how we turn a matrix into a word.

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix} \to 1\ 1\ 3\,3\ 3\ 1\ 2\ 2\ 2\ 2\ 3\ 1\ 1\ 1\ 2\ 3$$

We have 2 1s, then 0 2s, then 3 3s. Then we have 1 1, 4 2s, and 1 3. Continue like this.

Now we will proceed by applying the R-S bumping procedure to this word.

Example 1.4. Start with the previous word. Our partial outputs are

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \cline{1-3} \end{array}$$

and so on.

This gives us an $A \in \mathrm{SSYT}(\lambda, \bar{b})$. How do we make our corresponding recording tableau? The numbers that come from row i on the matrix get recorded in our recording tableau as i. We fill in the shape of the insertion tableau as the shape of the insertion tableau evolves.

Example 1.5. Our partial outputs for the recording tableau are

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} \end{array}$$

and so on.

1.3 Properties and relationship to representation theory

Theorem 1.1 (Knuth, c. 1980)**.** *RSK is a bijection.*

Proof. Here is what we need to show:

1. Φ is well-defined.
2. Φ^{-1} is well-defined.

Like before, we can prove these step by step and induct on the number of steps. $\square$

Theorem 1.2. *If $\Phi(M) = (A, B)$, then $\Phi(M^\top) = (B, A)$.*

From representation theory, we had the decomposition $M^\mu = \bigoplus_{|\lambda|=n} m_{\lambda,\mu} S^\lambda$. Here, $M^\mu = \mathrm{ind}_{S_{\mu_1}\times S_{\mu_2}\times\cdots}^{S_n} 1$. Take $\overline{a} = (a_1, \dots, a_k)$. We can define $M^{\overline{a}} = \mathrm{ind}_{S_{a_1}\times S_{a_2}\times\cdots}^{S_N} 1$, where $N = a_1 + \cdots + a_k$.

Theorem 1.3.
$$M^{\overline{a}} = \bigoplus_{|\lambda|=N} m_{\lambda,\overline{a}} S^\lambda,$$
where $m_{\lambda,\overline{a}} = \#\,\mathrm{SSYT}(\lambda, \overline{a})$.

Why should this be true?

$$\#M(\overline{a}, 1^N) = \frac{N!}{a_1!a_2!\cdots} = \dim(M^{\overline{a}}).$$

$$M(\overline{a}, 1^N) = \sum_{|\lambda|=N} \# \in SSYT(\lambda, \mu) \cdot \#\,\mathrm{SYT}(\lambda).$$

Now
$$M(\overline{a}, \overline{b}) = \langle \chi_{M^{\overline{a}}}, \chi_{M^{\overline{b}}} \rangle = \dim \mathrm{Hom}(M^{\overline{a}}, M^{\overline{b}}).$$